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Attractor States and Quantum Instabilities in de Sitter Space

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The asymptotic behavior of the energy–momentum tensor for a free quantized scalar field with mass m and curvature coupling ξ in de Sitter space is investigated. It is shown that for an arbitrary, homogeneous, and isotropic, fourth-order adiabatic state for which the two-point function is infrared finite, $\langle T_{ab} \rangle$ approaches the Bunch–Davies de Sitter invariant value at late times if $m^2 + \xi R > 0$. In the case $m = \xi = 0$, the energy–momentum tensor approaches the de Sitter invariant Allen–Folacci value for such a state. For $m^2 + \xi R = 0$ but m and ξ not separately zero, it is shown that at late times $\langle T_{ab} \rangle$ grows linearly in terms of cosmic time leading to an instability of de Sitter space. The asymptotic behavior is again independent of the state of the field. For $m^2 + \xi R < 0$, it is shown that, for most values of m and ξ, $\langle T_{ab} \rangle$ grows exponentially in terms of cosmic time at late times in a state dependent manner.

1. INTRODUCTION

The exponential expansion and maximal symmetry of de Sitter space allow for the possibility that quantum effects can be important even at late times when the universe is large. This has been born out by calculations of both the energy–momentum tensor $\langle T_{ab} \rangle$ and the quantity $\langle \phi^2 \rangle$. For example, it has been shown for free scalar fields that $\langle \phi^2 \rangle$ has a constant value and $\langle T_{ab} \rangle$ is equal to a constant times the metric tensor if the fields are in the de Sitter invariant state, which is sometimes referred to as the Euclidean vacuum and sometimes referred to as the Bunch–Davies state (Bunch and Davies, 1978; Chernikov and Tagirov, 1968; Dowker and Critchley, 1976; Tagirov, 1973). It has also been shown that the quantity $\langle \phi^2 \rangle$ diverges at late times in de Sitter space if $m^2 + \xi R \leq 0$, with m the mass of the

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field and ξ its coupling to the scalar curvature R (Allen and Folacci, 1987; Linde, 1982; Starobinsky, 1982; Vilenkin and Ford, 1982).

The fact that quantum effects can be significant when the universe is large means that it is possible for backreaction effects to also be important. In fact, if certain components of the energy–momentum tensor become too large then de Sitter space will be unstable. This is because the backreaction of the fields on the spacetime geometry will cause the expansion rate to cease being exponential. It is well known that such instabilities occur for certain classical scalar fields in de Sitter space (Dolgov, 1983; Ford, 1987) and for certain interacting quantized fields (Ford, 1987). It would seem likely that similar instabilities might occur for free quantized fields particularly given the divergent behavior exhibited by the quantity $\langle\phi^2\rangle$ in some cases.

Most previous studies of quantum effects in de Sitter space have focused on either de Sitter invariant states or the special O(4) invariant state discovered by Allen (1985) that occurs for the massless minimally coupled scalar field. Some exceptions are studies of the behavior of the quantity $\langle\phi^2\rangle$ for arbitrary states (Linde, 1982; Starobinsky, 1982; Vilenkin and Ford, 1982), and a study of the energy of excited states for scalar fields (Redmount, 1989). The most general class of states for which the energy–momentum tensor is ultraviolet finite in a homogeneous and isotropic spacetime are fourth-order adiabatic states (See for example, Birrel and Davies, 1982). It is important to consider this general class of states because, unless the universe was expanding exponentially when it began, it is very unlikely that the fields will be in de Sitter invariant states.

In this paper, we investigate the asymptotic behavior of $\langle T_{ab}\rangle$ for quantized scalar fields in arbitrary fourth-order adiabatic states in de Sitter space. The wave equation for free scalar fields can be solved analytically in de Sitter space for all values of the mass and the curvature coupling. Its solutions depend only on the wave number k of the mode and the parameter $\nu^2 = \frac{9}{4} - m^2\alpha^2 - 12\xi$, with $R = 12\alpha^{-2}$ the constant scalar curvature of de Sitter spacetime. For $\Re(\nu) < \frac{3}{2}$, corresponding to $m^2 + \xi R > 0$, we prove that for all fourth-order adiabatic states the renormalized value of $\langle T_{ab}\rangle$ at late times asymptotically approaches the value it has if the field is in the Bunch–Davies state. The conformally invariant scalar field ($m = 0, \xi = \frac{1}{6}$) falls into this class.

The case $\nu = \frac{3}{2}$ corresponding to $m^2 + \xi R = 0$ is more complicated. In the massless minimally coupled case we prove that $\langle T_{ab}\rangle$ for all physically admissible states approaches the Allen–Folacci de Sitter invariant value (Allen and Folacci, 1987; Folacci, 1991a,b; Kirsten and Garriga, 1993). Numerical evidence for this result was found previously in Habib *et al.* (2000). If $m^2 = -\xi R \neq 0$ then we show that $\langle T_{ab}\rangle$ grows linearly in terms of cosmic (proper) time at late times. This leads to an instability of de Sitter space.

An instability also occurs for most values of m and ξ if $\nu > \frac{3}{2}$, corresponding to $m^2 + \xi R < 0$. In these cases $\langle T_{ab} \rangle$ grows exponentially at late times for all fourth-order adiabatic states in a state dependent manner.

The paper is organized as follows: In Section 2 we review the quantization of free scalar fields in a general Robertson-Walker (RW) spacetime. In Section 3 we analyze the late time behavior of $\langle T_{ab} \rangle$ in de Sitter space for the case $\nu < 3/2$. The cases $\nu = 3/2$ and $\nu > 3/2$ are discussed in Sections 4 and 5, respectively. A brief discussion of our results is given in Section 6.

2. SCALAR FIELD IN A ROBERTSON-WALKER BACKGROUND

The metric for a general RW spacetime can be written in the form

$$ds^2 = a^2(\eta)\left(-d\eta^2 + \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2\right). \tag{2.1}$$

Here η is the conformal time, $a(\eta)$ is the scale factor, and $\kappa = 0, +1, -1$ corresponds to the cases of flat, spherical, and hyperbolic spatial sections, respectively. Throughout we use units such that $\hbar = c = 1$ and the Misner *et al.* (1973) conventions for the curvature tensors, $R^a_{bcd} = \Gamma^a_{bd,c} - \ldots$ and $R_{ab} = R^c_{acb}$.

We consider in this paper a free quantized scalar field ϕ with the quadratic action

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}[(\nabla_a\phi)g^{ab}(\nabla_b\phi) + m^2\phi^2 + \xi R\phi^2], \tag{2.2}$$

where ∇_a denotes the covariant derivative, R is the scalar curvature, and $g \equiv \det(g_{ab})$. The mass m and curvature coupling ξ are allowed to have any real value. The wave equation for ϕ obtained by varying this action is

$$[-\Box + m^2 + \xi R]\phi(\eta, \mathbf{x}) = \left[\frac{1}{a^4}\frac{\partial}{\partial\eta}\left(a^2\frac{\partial}{\partial\eta}\right) - \frac{1}{a^2}\Delta^{(3)} + m^2 + \xi R\right]\phi = 0, \tag{2.3}$$

with $\Delta^{(3)}$ the covariant spatial Laplacian. For spacetimes with the metric (2.1) the field ϕ can be expanded as a mode sum in the form (Birrell and Davies, 1982)

$$\phi(\eta, \mathbf{x}) = \frac{1}{a(\eta)}\int d\tilde{\mu}(\mathbf{k})[a_{\mathbf{k}}Y_{\mathbf{k}}(\mathbf{x})\psi_k(\eta) + a^\dagger_{\mathbf{k}}Y^*_{\mathbf{k}}(\mathbf{x})\psi^*_k(\eta)], \tag{2.4}$$

where the integration measure is given by

$$\int d\tilde{\mu}(\mathbf{k}) \equiv \begin{cases} \int d^3\mathbf{k} & \text{if } \kappa = 0, \\ \int_0^\infty dk \sum_{l,m} & \text{if } \kappa = -1, \\ \sum_{k,l,m} & \text{if } \kappa = +1, \end{cases}$$

and the spatial part of the mode functions $Y_{\mathbf{k}}(\mathbf{x})$ obeys the equation

$$-\Delta^{(3)}Y_{\mathbf{k}}(\mathbf{x}) = (k^2 - \kappa)Y_{\mathbf{k}}(\mathbf{x}), \tag{2.5}$$

with $k = 1, 2, \ldots,$ in the case of closed spatial sections, $\kappa = +1$. The time dependent part of the mode functions ψ_k obeys the equation

$$\psi_k'' + \left[k^2 + m^2a^2 + \left(\xi - \frac{1}{6}\right)a^2R\right]\psi_k = 0, \tag{2.6}$$

where primes denote derivatives with respect to the conformal time variable η, and the scalar curvature in a general RW spacetime is given by

$$R = 6\left(\frac{a''}{a^3} + \frac{\kappa}{a^2}\right). \tag{2.7}$$

For the quantum field to satisfy the canonical commutation relations, the creation and annihilation operators are required to obey the commutation relations $[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}$, whereupon the ψ_k must obey the Wronskian condition

$$\psi_k\psi_k^{*\prime} - \psi_k^*\psi_k' = i. \tag{2.8}$$

The unrenormalized expressions for the components of $\langle T_{ab}\rangle$ are given by Bunch (1980)

$$\begin{aligned}\varepsilon_u = -\left\langle T_0^0\right\rangle_u = \frac{1}{4\pi^2a^4}\int d\mu(k)(2n_k+1)\bigg\{&|\psi_k'|^2 + (k^2+m^2a^2)|\psi_k|^2\\ &+ (6\xi-1)\left[\frac{a'}{a}(\psi_k\psi_k^{*\prime} + \psi_k^*\psi_k') - \left(\frac{a'^2}{a^2} - \kappa\right)|\psi_k|^2\right]\bigg\},\end{aligned} \tag{2.9a}$$

$$\begin{aligned}-\varepsilon_u + 3p_u = \langle T\rangle_u = \frac{1}{2\pi^2a^4}\int d\mu(k)(2n_k+1)\bigg\{&-m^2a^2|\psi_k|^2 + (6\xi-1)\\ &\times\left[-|\psi_k'|^2 + \frac{a'}{a}(\psi_k\psi_k^{*\prime} + \psi_k^*\psi_k')\right] + (6\xi-1)\\ &\times\left[k^2 + m^2a^2 + \left(\frac{a''}{a} - \frac{a'^2}{a^2}\right) + \left(\xi - \frac{1}{6}\right)a^2R\right]|\psi_k|^2]\bigg\}.\end{aligned} \tag{2.9b}$$

where we are considering states with an arbitrary number of particles $n_k = \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\rangle$, and the scalar measure $d\mu(k)$ is given by

$$\int d\mu(k) \equiv \begin{cases}\int_0^\infty dk\, k^2 & \text{if } \kappa = 0, -1,\\ \sum_1^\infty k^2 & \text{if } \kappa = +1.\end{cases}$$

As we are considering spatially homogeneous and isotropic initial states (consistent with the RW symmetry), n_k depends only on the magnitude k of the spatial wave vector $\mathbf{k}$.

Since $\langle T_{ab}\rangle_u$ is quartically divergent, a procedure for defining finite, renormalized expectation values must be given. We will follow the adiabatic regularization method (Fulling and Parker, 1974; Fulling *et al.*, 1974; Parker, 1966; Parker and Fulling, 1974). In this method the renormalization counterterms are constructed using a fourth-order expansion for $\langle T_{ab}\rangle$. We denote these counterterms by $\langle T_{ab}\rangle_{ad}$. They are given in Anderson and Parker (1987) and Bunch (1980). The renormalized expressions are then

$$\langle T_{ab}\rangle_{\rm ren} = \langle T_{ab}\rangle_u - \langle T_{ab}\rangle_{ad}. \tag{2.10}$$

This subtraction scheme is not manifestly covariant in form, since space and time are treated quite differently. However, adiabatic regularization is equivalent to a covariant point splitting procedure in which the points are split only in the spacelike hypersurface of constant η (Anderson and Parker, 1987; Birrell, 1978), and the *values* of the renormalized $\langle T_{ab}\rangle$ obtained by this procedure are the same as in a strictly covariant one. Hence this subtraction procedure does correspond to adjustment of counterterms to the quantum effective action, and $\langle T_{ab}\rangle_{\rm ren}$ is covariantly conserved. As discussed in detail in Anderson and Parker (1987), the adiabatic terms in all cases consist of an integral rather than a sum over k. The reason is that subtraction corresponds to purely local counterterms in the effective action, and thus must be independent of the global compactness or noncompactness of the spatial sections.

A useful variation of the method of adiabatic regularization has been developed by two of us (Anderson and Eaker, 2000). In this method one first computes a quantity $\langle T_{ab}\rangle_d$, obtained by expanding the adiabatic counterterms $\langle T_{ab}\rangle_{ad}$ in inverse powers of k and truncating at order k^{-3}. The same renormalized energy–momentum tensor defined in Eq. (2.10) is separated into the sum of two *finite* terms by adding and subtracting $\langle T_{ab}\rangle_d$ so that

$$\begin{aligned}\langle T_{ab}\rangle_{\rm ren} &= \langle T_{ab}\rangle_n + \langle T_{ab}\rangle_{an},\\ \langle T_{ab}\rangle_n &= \langle T_{ab}\rangle_u - \langle T_{ab}\rangle_d,\\ \langle T_{ab}\rangle_{an} &= \langle T_{ab}\rangle_d - \langle T_{ab}\rangle_{ad}.\end{aligned} \tag{2.11}$$

The full expressions for $\langle T_{ab}\rangle_d$ and $\langle T_{ab}\rangle_{an}$ are given in Anderson and Eaker (2000) for a general RW spacetime. The advantage of this splitting is that $\langle T_{ab}\rangle_n$ and $\langle T_{ab}\rangle_{an}$ are separately conserved, and moreover, $\langle T_{ab}\rangle_{an}$ may be computed analytically in terms of the scale factor $a(\eta)$ and its derivatives (Anderson and Eaker, 2000). Thus the state dependence of the renormalized $\langle T_{ab}\rangle_{\rm ren}$ resides completely in $\langle T_{ab}\rangle_n$, which can be computed numerically.

3. $\Re(\nu) < 3/2$

We now focus on the asymptotic evaluation of $\langle T_{ab}\rangle$ in de Sitter space. The geometry of de Sitter spacetime can de described in a number of different coordinate systems. If $\kappa = 0$, the spatial sections are flat and the scale factor is

$$a(\eta) = -\frac{\alpha}{\eta}, \quad -\infty < \eta < 0, \quad \kappa = 0, \tag{3.1}$$

with α a real, positive constant, and $R = 12\alpha^{-2}$. If $\kappa = +1$ then the scale factor is

$$a(\eta) = \alpha \sec\eta, \quad -\frac{\pi}{2} < \eta < \frac{\pi}{2}, \quad \kappa = +1, \tag{3.2}$$

which is equivalent to $a(\eta) = \alpha \csc\eta$ with $0 < \eta < \pi$. Again $R = 12\alpha^{-2}$.

We shall use the $\kappa = 0$ coordinates in the analysis of the $\Re(\nu) < 3/2$ case and the $\kappa = 1$ coordinates for the cases $\nu \geq 3/2$. No confusion should be caused by our use of the same symbol η for conformal time in both cases of flat and closed spatial sections, since these are treated separately.

For the case of Eq. (3.1) the general solution to the mode equation can be written as (Bunch and Davies, 1978)[5]

$$\psi_k(\eta) = \frac{1}{2}(-\pi\eta)^{\frac{1}{2}}\, e^{\frac{i\nu\pi}{2}} \left[c_1(k)H_\nu^{(1)}(-k\eta) + c_2(k)H_\nu^{(2)}(-k\eta)\right], \tag{3.3}$$

where the $H_\nu^{(1),(2)}$ are Hankel functions and

$$\nu^2 \equiv \frac{9}{4} - m^2\alpha^2 - 12\xi. \tag{3.4}$$

When $\nu^2 > 0$ we will choose ν to be the positive root of (3.4). From Eq. (3.3) we see that solutions to the mode equation in de Sitter space depend on m and ξ only through their dependence on the parameter ν. Note that because of the minus sign in the arguments of the Hankel functions, it is the function $H_\nu^{(1)}$ that corresponds to a positive frequency mode in the large k limit. The normalization of the mode function in (3.3) has been chosen so that the Wronskian condition (2.8) becomes simply

$$|c_1(k)|^2 - |c_2(k)|^2 = 1. \tag{3.5}$$

The Bunch–Davies state is defined by the choice, $c_1 = 1$ and $c_2 = 0$ (with $n_k = 0$) for all k. The renormalized value of $\langle T_{ab}\rangle$ in the Bunch–Davies state is (Bunch and Davies, 1978; Dowker and Critchley, 1976)

$$\langle T_{ab}\rangle_{\rm BD} = -\frac{g_{ab}}{64\pi^2}\left\{m^2\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\left[\psi\left(\frac{3}{2}+\nu\right) + \psi\left(\frac{3}{2}-\nu\right)\right.\right.$$

[5] In Bunch and Davies (1978), the arguments of the Hankel functions are given as $k\eta$ rathen than $-k\eta$. We have chosen to use nonnegative arguments to avoid complications that result from the fact that these functions have branch cuts along the negative real axis.

$$- \log\left(\frac{12m^2}{R}\right)\Bigg] - m^2\left(\xi - \frac{1}{6}\right) R - \frac{1}{18} m^2 R$$

$$- \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R^2 + \frac{R^2}{2160}\Bigg\}, \tag{3.6a}$$

where $\psi(z) = \frac{d \log \Gamma(z)}{dz}$ is the digamma function.

For the general state with $c_2 \neq 0$ to remain fourth-order adiabatic, we must have for large values of k

$$c_2(k) = \frac{C(k)}{k^4}, \tag{3.7}$$

for some complex function $C(k)$ which vanishes in the limit $k \to \infty$. This condition is necessary for an arbitrary (spatially homogeneous) state to posses a finite energy–momentum tensor after the fourth-order adiabatic subtraction defined by (2.10). Likewise the same condition of finite $\langle T_{ab} \rangle$ requires us to restrict the average number of particles $\langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \rangle = n_k$ by

$$n_k = \frac{N(k)}{k^4}, \tag{3.8}$$

for some real function $N(k)$ which vanishes in the limit $k \to \infty$. The two ultraviolet conditions

$$\lim_{k\to\infty} |C(k)| = \lim_{k\to\infty} N(k) = 0, \tag{3.9}$$

on the physically allowed states guarantee that the Green's function for the scalar field is locally of the Hadamard form (Junker, 1995, Linding, 1999; Lüders and Roberts, 1999; Najmi and Ottewill, 1985), and that the divergences of $\langle T_{ab} \rangle$ match those of the fourth-order adiabatic vacuum, and are removed by the adiabatic subtraction procedure.

To understand why the Bunch–Davies state serves as an attractor state let us observe that at late times $\eta \to 0^-$, the general state mode function (3.3) behaves like

$$\psi_k \sim (-\eta)^{\frac{1}{2} - \nu} \sim a^{\nu - \frac{1}{2}}. \tag{3.10}$$

Substituting this into (2.9a) and (2.9b) shows that to leading order at late times the contributions to the mode sums of $\langle T^a_b \rangle_u$ behave like $(-\eta)^{3-2\nu} \sim a^{2\nu-3}$ for ν real. Since the renormalization counterterms are state independent (Bunch, 1980), the state dependent terms are the same in the unrenormalized and renormalized quantities. One can perform all the UV renormalization in the Bunch–Davies state at a fixed time and collect the remaining finite state dependent terms which are unaffected by the subtraction procedure, and they all fall off at least as fast as $(-\eta)^{3-2\nu}$ as $\eta \to 0^-$ for $\Re(\nu) < \frac{3}{2}$.

To prove this result we first note that for an arbitrary fourth-order adiabatic state we can make use of Eq. (3.5) to show that

$$\langle T_{ab}\rangle_{\rm ren} = \langle T_{ab}\rangle_{\rm BD} + \langle T_{ab}\rangle_{\rm SD}, \tag{3.11}$$

where $\langle T_{ab}\rangle_{\rm SD}$ is composed of finite state dependent terms, depending on the coefficients $c_1(k)$, $c_2(k)$, and n_k. It may be expressed as an integral over the wave number k in the form

$$\langle T_{ab}\rangle_{\rm SD} = \frac{1}{4\pi^2}\int_0^\infty {\rm d}k\, I_{ab}(k,\eta). \tag{3.12}$$

The leading order contributions to I^a_b at late times are the same as those for $\langle T^a_b\rangle_u$ discussed above. Thus they go like $(-\eta)^{3-2\nu}$. To find the asymptotic behavior of $\langle T^a_b\rangle_{\rm SD}$ one must first compute the mode integral and then take the limit $\eta \to 0^-$. We proved in Anderson *et al.* (2000) that it is possible to interchange the order of these operations. Since I^a_b vanishes at late times for all values of k it is then clear that $\langle T^a_b\rangle_{\rm SD}$ do as well. Therefore, for an arbitrary fourth-order adiabatic state and for $\Re(\nu) < 3/2\langle T^a_b\rangle$ asymptotically approaches the values they would have if the field was in the Bunch–Davies state.

4. $\nu = 3/2$

To treat the case $\nu = 3/2$ carefully, it is easiest to work with closed spatial sections and a discrete set of mode functions in order to treat the most infrared sensitive, spatially homogeneous $k = 1$ mode separately from the rest, instead of dealing with an infrared sensitive continuous mode integral. The scale factor for $\kappa = +1$ is given by Eq. (3.2). The general solution of the mode equation (2.6) is

$$\psi_k(\eta) = \alpha_k\, f_k(\eta) + \beta_k\, f_k^*(\eta), \tag{4.1}$$

with

$$f_k(\eta) = \frac{e^{-ik\eta}}{[2k(k^2-1)]^{\frac{1}{2}}}(k + i\ \tan\eta), \quad k = 2, 3, \ldots, \tag{4.2}$$

The Wronskian condition

$$f_k\, f_k^{*\prime} - f_k^*\, f_k^{\prime} = i, \tag{4.3}$$

gives

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \tag{4.4}$$

The Bunch–Davies state is given by $\alpha_k = 1$ and $\beta_k = 0$.

In Eq. (4.2), the $k = 1$ mode function is singular. Thus it must be treated separately if the two-point function is to be free of infrared divergences. The behavior of the $k = 1$ mode is

$$\psi_1(\eta) = \sec\eta\left[\frac{A}{2}(\eta + \sin\eta\,\cos\eta) + B\right] \tag{4.5}$$

and its normalization is

$$A^*B - B^*A = i. \tag{4.6}$$

It has been shown that if $\nu = 3/2$ then the quantity $\langle\phi^2\rangle$ grows linearly in terms of cosmic time t with $dt = a d\eta$ (Allen and Folacci, 1987; Linde, 1982; Vilenkin and Ford, 1982). The same type of behavior occurs for the nonzero components of $\langle T^a_b\rangle$ if $m^2\alpha^2 = -12\xi \neq 0$. To see this one can use Eq. (2.11) to divide the energy–momentum tensor into a state dependent part $\langle T_{ab}\rangle_n$ and a state independent part $\langle T_{ab}\rangle_{an}$. They are separately conserved and explicit expressions for them in a general RW spacetime are given in Anderson and Eaker (2000). The quantity $\langle T_{ab}\rangle_n$ can be computed by substituting Eqs. (4.1), (4.2), and (4.5) into Eqs. (2.9a) and (2.9b) and subtracting the relevant expressions for $\langle T_{ab}\rangle_d$ that are given in Anderson and Eaker (2000). We find that $\langle T_{ab}\rangle_n$ approaches a state dependent constant in the limit $\eta \to \pi/2$. We also find that the quantity $\langle T_{ab}\rangle_{an}$ has the asymptotic behavior

$$\langle T_{ab}\rangle_{an} \to g_{ab}\frac{3\xi}{4\pi^2\alpha^4}\,\log(\mu a) \to g_{ab}\frac{3\xi t}{4\pi^2\alpha^5} \tag{4.7}$$

Thus the nonzero components of $\langle T^a_b\rangle_{\rm ren}$ diverge in a state independent manner as $\eta \to \pi/2$.

In the important case that $m = \xi = 0$, two surprising results occur. First from Eq. (4.7) it is seen that $\langle T_{ab}\rangle_{an}$ does not diverge asymptotically. Second $\langle T_{ab}\rangle_n$ does not approach a state dependent constant. To see why the latter result occurs, it is useful to introduce the de Sitter invariant energy–momentum tensor found by Allen and Folacci (Allen and Folacci, 1987; Folacci, 1991a,b; Kirsten and Garriga, 1993). One can derive the expression for their energy–momentum tensor by not including the $k = 1$ mode in the mode sum, choosing the Bunch–Davies state, $\alpha_k = 1$ and $\beta_k = 0$ for the modes with $k > 1$, and substituting the resulting expressions into Eq. (2.10). The result is

$$\langle T_{ab}\rangle_{\rm AF} = g_{ab}\frac{119R^2}{138240\pi^2}. \tag{4.8}$$

Using Eqs. (4.4), (2.9a), and (2.9b) we then find that for a general state (including the contribution of the $k = 1$ mode)

$$\varepsilon = -\langle T^0_{\ 0}\rangle_{\rm ren}$$
$$= -\langle T^0_{\ 0}\rangle_{\rm AF} + (1 + 2n_1)\frac{|A|^2\cos^6\eta}{\pi^2\alpha^4} + \frac{1}{4\pi^2\alpha^4}\sum_{k=2}^{\infty}\Big\{[2n_k + 2(1 + 2n_k)$$

$$\times\,|\beta_k|^2]\left[k^3\cos^4\eta + k\left(-\cos^4\eta + \frac{1}{2}\cos^2\eta\right)\right] + (1+2n_k)$$
$$\times\left[(\beta_k\alpha_k^*\,e^{2ik\eta} + \beta_k^*\alpha_k\,e^{-2ik\eta})k\left(-\cos^4\eta + \frac{1}{2}\cos^2\eta\right)\right.$$
$$\left.\left. + i(\beta_k\alpha_k^*\,e^{2ik\eta} - \beta_k^*\,e^{-2ik\eta})k^2\cos^3\eta\,\sin\eta\right]\right\}, \tag{4.9a}$$

$$\langle T\rangle_{\mathrm{ren}} = \langle T\rangle_{\mathrm{AF}} + (1+2n_1)\frac{2|A|^2\cos^6\eta}{\pi^2\alpha^4} - \frac{1}{4\pi^2\alpha^4}\sum_{k=2}^{\infty}\{[2n_k + 2(1+2n_k)$$
$$\times\,|\beta_k|^2]\,k\cos^2\eta + (1+2n_k)[(\beta_k\alpha_k^*\,e^{2ik\eta} + \beta_k^*\alpha_k\,e^{-2ik\eta})(-2k^3\cos^4\eta$$
$$+ k\cos^2\eta) + 2i(\beta_k\alpha_k^*\,e^{2ik\eta} - \beta_k^*\alpha_k\,e^{-2ik\eta})k^2\cos^3\eta\,\sin\eta\}. \tag{4.9b}$$

Provided the k sums converge, it is clear that all the state dependent terms contain at least one factor of $a^{-2} = \alpha^{-2}\cos^2\eta$, and so vanish in the limit of $\eta \to \frac{\pi}{2}$. However, the requirement that the state be fourth-order adiabatic just guarantees this convergence, for the same reason as in the previous analysis in spatially flat coordinates. Indeed we have

$$|\beta_k| = \frac{C(k)}{k^4},$$
$$n_k = \frac{N(k)}{k^4}, \tag{4.10}$$

for some $C(k)$ and $N(k)$ that vanish as $k \to \infty$. This is sufficient to guarantee the absolute convergence of all terms in the sums. Since all state dependent terms are multiplied by at least two powers of $\cos\eta = \alpha/a$, which vanishes in the late time limit $\eta \to \frac{\pi}{2}$, we conclude that any fourth-order adiabatic state of the massless, minimally coupled scalar field for which the two-point function is infrared finite, has an energy–momentum tensor which approaches the AF value, $\langle T_{ab}\rangle_{\mathrm{AF}}$ in the late time limit $\eta \to \frac{\pi}{2}$.

5. $\nu > 3/2$

For $\nu > 3/2$, we again use the $\kappa = 1$ coordinates. The mode functions are of the form (4.1) with the normalization (4.4). In Anderson *et al.* (2000), it was shown that

$$f_k(\eta) = \left[\frac{\Gamma\left(k+\frac{1}{2}+\nu\right)\Gamma\left(k+\frac{1}{2}-\nu\right)}{2}\right]^{\frac{1}{2}}\frac{e^{-ik\eta}}{k!}$$
$$\times\,F\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu;k+1;\frac{1-i\tan\eta}{2}\right), \tag{5.1}$$

where F is the hypergeometric function. For all real values of ν

$$f_k\left(\eta \to \frac{\pi}{2}\right) \to \left[\frac{\Gamma\left(k+\frac{1}{2}-\nu\right)}{2\Gamma\left(k+\frac{1}{2}+\nu\right)}\right]^{\frac{1}{2}} \frac{\Gamma(2\nu)}{\Gamma\left(\frac{1}{2}+\nu\right)} \frac{(-i)^k}{k!} \left(\frac{i\ \sec\eta}{2}\right)^{\nu-\frac{1}{2}}. \quad (5.2)$$

Thus the modes grow like $a^{\nu-\frac{1}{2}}$ at late times which implies that the leading order terms in $\langle T_b^a \rangle$ grow like $a^{2\nu-3}$ at late times. Therefore $\langle T_b^a \rangle$ diverges exponentially in terms of the cosmic time, t, in a state dependent manner unless the leading order terms cancel. In Anderson *et al.* (2000), it was shown that this occurs for the following values of m and ξ for a given value of ν

$$m^2\alpha^2 = -\frac{\nu(2\nu-3)(2\nu-1)}{4(\nu-2)},$$

$$\xi = \frac{(2\nu-3)}{8(\nu-2)}. \quad (5.3)$$

The next to leading order terms in $\langle T_b^a \rangle$ go like $a^{2\nu-5}$ so if $\nu > 5/2$, $\langle T_b^a \rangle$ still diverges exponentially unless the coefficient of the next to leading order terms also vanishes.

6. SUMMARY AND DISCUSSION

We have shown that in the case $\Re(\nu) < 3/2$ the Bunch–Davies state serves as a fixed point attractor for the energy–momentum tensor in the sense that, for an arbitrary fourth-order adiabatic state, $\langle T_{ab} \rangle$ approaches the value it would have if the field was in the Bunch–Davies state. This is a striking result. Certainly no such attractor behavior of $\langle T_{ab} \rangle$, independent of initial conditions occurs in Minkowski space for any mass. One may regard this result as a kind of cosmic "no hair" theorem for scalar quantum fields in de Sitter space. It is in accord with one's classical intuition that any initial energy density satisfying the weak energy condition ($\varepsilon + p > 0$) is redshifted away by the exponential de Sitter expansion. At asymptotically late times what is left behind is a kind of frozen "quantum vacuum energy condensate," satisfying the de Sitter invariant equation of state $p = -\varepsilon$. This result justifies the choice of the Bunch–Davies vacuum in calculations of quantum fluctuations of *free* fields, i.e., without backreaction, in a long-lived de Sitter expansion phase of inflationary cosmological models.

For $\nu = 3/2$, we have shown that if $m^2\alpha^2 = -12\xi \neq 0$, the nonvanishing components of $\langle T_b^a \rangle$ diverge linearly with respect to cosmic time at late times indicating the existence of a quantum instability. For the case $m = \xi = 0$, the energy–momentum tensor does not asymptotically diverge. Instead, for any fourth-order adiabatic state for which the two-point function is infrared finite, $\langle T_b^a \rangle$ asymptotically approaches the de Sitter invariant value found by Allen and Folacci (Allen and Folacci, 1987; Folacci, 1991a,b; Kirsten and Garriga, 1993). There are two reasons for this surprising result. One is that the coefficient of the asymptotically divergent terms in $\langle T^a_b \rangle$ vanishes if $m = \xi = 0$. The other is that the coefficient of the leading order (at late times) mode contributions to $\langle T^a_b \rangle$ also vanishes if $m = \xi = 0$.

Finally, for the case $\nu > 3/2$ we find that, for most values of m and ξ, the nonzero components of $\langle T^a_b \rangle$ diverge exponentially in proper time at late times in a state dependent manner for an arbitrary fourth-order adiabatic state.

The divergent behavior of the energy–momentum tensor found for $\nu > 3/2$ is of exactly the same type as that found for the energy–momentum tensor of classical scalar fields with the same values of m and ξ (Dolgov, 1983; Ford, 1987). However, the fact that the effective mass of the field, which is equal to $m^2\alpha^2 + \xi R$, is tachyonic in this case and the fact that either m^2 or ξ must be negative means that the resulting instability of de Sitter space is probably of little physical relevance. A similar observation applies to the instability found for $\nu = 3/2$. Here the effective mass is zero, but it is still necessary to have a negative value for either m^2 or ξ. However, it is possible that a similar divergence of the energy–momentum tensor for gravitons occurs in de Sitter space. The reason is that in any RW spacetime the mode equation for gravitons in a particular gauge is identical to that for the massless minimally coupled scalar field (Grishchuk, 1974). Work is currently in progress to calculate the energy–momentum tensor for gravitons in de Sitter space in order to determine if such an instability exists.

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